From Music to Mathematics: Exploring the Connections

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Recommended Music to Accompany Chapter 5
Musical Group Theory

The selections below include some type of mathematical symmetry: translation (transposition), vertical reflection (retrograde), horizontal reflection (inversion), or some combination of these symmetries. Along with 180° rotation, these symmetries form a special musical subgroup of a well known mathematical group called the dihedral group of degree 4 (symmetries of the square; see Section 5.3 of the text). Some composers use these ideas in music quite intentionally to generate new thematic material out of an original idea. Bach was certainly one of the great masters of this, although it is important to note that he did so while never losing site of the musical art in his creations.

What to listen for: Excerpts for many of the pieces described below can be found in the textbook. It is important to follow the score as you listen, as this helps identify the composer's intent and reveal the symmetry transformation. It is also a good way to engage your brain while you listen. Try to hear where a retrograde or an inversion takes place in the music. As with the recommended music for Chapter 2, this collection contains a wide variety of musical styles and various tonalities. Try to discern the differences between earlier and later forms of composition as well as between different forms of music (fugue, oratorio, symphony, sonata, song, etc.) The music is organized by symmetry types, which are underlined and highlighted in boldface.

1. Ludwig Van Beethoven, *Symphony No. 5 in C minor*, 1st movement, Op. 67, 1807–1808. Track 1, Disc 3 from the complete collection of Beethoven's Nine Symphonies (he actually wrote 10, but the 10th was unfinished) performed by the Chicago Symphony Orchestra under the direction of Sir Georg Solti. This famous symphony took the idea of elaborating upon a simple motif to new heights. The opening "da da da dum" motif, once called "fate knocking at the door" by Beethoven himself, is **transposed**, inverted, elongated, shortened, and continually repeated throughout the work to create the bulk of the music (see Figure 5.10 in the textbook). The entire first minute of this symphony consists solely of this famous motif. It is worth listening to the entire symphony to see how Beethoven brings the motif back unexpectedly in the final movement.

2. Samuel Barber, *Adagio for Strings*, 1936. Track 8 on a Sony Classical CD featuring the New York Philharmonic, conducted by Leonard Bernstein (no. 39 of 100 in the Royal Edition series). This is the orchestral version of the piece "Agnus Dei," one of the recommended works to accompany Chapter 2. Barber, one of the most well-known twentieth-century composers in the U.S., took the slow movement from his String Quartet in B minor, Op. 11, 1936, and reworked it into this hauntingly beautiful work for string orchestra. This was the music used for Oliver Stone's great Vietnam War film *Platoon*. Much later in his career, Barber set the music to the traditional Latin "Agnus Dei" text. The piece is in a minor key and drips with luscious harmonies and a tantalizingly slow, somber melody that appears to be continuously climbing up the $B^\flat$ minor scale (see Figure 5.11 of the course textbook). The melody features a **transposition** of a three-note motif that remains within the given key. Barber's work has often been described as neo-romantic, harkening back to the great Romantic composers (e.g., Beethoven and Brahms.)

3. Kurt Weill, *Mack the Knife* (lyrics by Bertolt Brecht), 1928. Track 16 on a Polygram Records CD titled The *Complete* Ella in Berlin: Mack the Knife, featuring the great jazz singer Ella Fitzgerald recorded live in Berlin in 1960. In her classic rendition of this well-known song, where Ella forgets the lyrics midway through the song, each successive verse is **transposed** up a half step into a new key. This standard translation is a great method of keeping the listener interested, particularly if other aspects of the song are falling apart. Unable to recall the correct lyrics, Ella did what many great jazz musicians do—she improvised. Her memorable rendition earned her two Grammy awards in 1961.

4. Guillaume de Machaut (c. 1300 - 1377), *Ma Fin Est Mon Commencement* ("My End is My Beginning"). Track 19 from a Hyperion CD titled Messe De Notre Dame performed by the outstanding Hilliard Ensemble. As the text indicates, this work is a **retrograde** canon in music and almost in words as well. This is one of the earliest known examples of a retrograde in music that was actually written down. The piece is apparently sung using Pythagorean tuning. This is particularly recognizable near the cadences or ends of each phrase. The singers are keeping as many perfect fifths as possible in a 3:2 ratio to produce the Pythagorean tuning using their voices.

5. George Frideric Handel, *Hallelujah Chorus* from *Messiah*, 1741. Track 18, Disc 2 from a BMG Entertainment CD titled Messiah performed by Musica Sacra, directed by Richard Westenburg. Arguably Handel's most famous work, *Messiah* is a grand oratorio telling the story of Jesus that Handel obsessively composed in only 24 days. The text is a compilation of verses from the Bible. Upon finishing the now famous Hallelujah Chorus, Handel exclaimed, "I did think I did see all Heaven before me, and the Great God Himself!" Early on in the chorus (about 22 seconds in) we hear all the voices in unison singing "for the Lord God Omnipotent reigneth," a musical phrase which, if one neglects duration, is in **retrograde**. This motif rises from scale degree 1 to 4 and then returns, a musical palindrome given by the notes D E F♯ G (G) G F♯ E D. The parenthetical G is an octave lower than its surrounding Gs. Although the rhythm is not quite a perfect retrograde, Handel is using vertical symmetry to proclaim the brilliance and balance of the "Lord God." To emphasize this significance, the motif appears repeatedly, often in different voices, throughout the piece.

6. Johann Sebastian Bach, *Crab Canon*, piece no. 9 from the *Musical Offering*, BWV 1079, 1747. Track 17 off the Naxos CD J. S. Bach: The Musical Offering, performed by the Capella Istropolitana under the direction of Christian Benda. This is a fantastic example of a perfect **retrograde** (a musical palindrome) over 18 measures of music (see Figure 5.14 in the text). The top voice announces the Royal theme in the first nine measures and then follows with an elaborate eighth-note countermelody over the next nine measures. Not only does this entire 18-measure part sound great when played backward, but it works forward and backward at the same time! While the top voice plays its part, the bottom voice is playing the same part in retrograde. Even though they are traveling in opposite directions, the two voices blend beautifully. Mathematically, the interchange and reflection can best be visualized on a Möbius strip. Be sure to watch the wonderful YouTube video of this construction by Jos Leys, available at `https://www.youtube.com/watch?v=xUHQ2ybTejU`

7. Joseph Haydn, *Piano Sonata in A Major, "Minuet and Trio"*, Hob. XVI/26 or Landon 41, 1773. Track 13, Disc 6 from the complete collection of Haydn's piano sonatas performed by John McCabe (Decca Record Company Limited, London). The opening minuet is titled "Menuet Al Rovescio" (Minuet in Reverse) and is an exact musical palindrome (**retrograde**) with the vertical reflection

occurring at the end of measure ten (see Figure 5.12 in the course text). Note that McCabe even reverses the trill (a quick flourish of extraneous notes) in bars eight and nine. The $\frac{3}{4}$ meter seems crucial to making the retrograde work harmonically as Haydn often emphasizes two of the three notes of the underlying chord in beats one and three of each measure. Thus, on playing such a measure backwards, the chord and corresponding harmony remain unchanged. Haydn first composed his clever musical palindrome for his *Symphony No. 47 in G major* (1772), appropriately called, "The Palindrome." He later used it in his *Sonata No. 4 for Piano and Violin.*

8. Johann Sebastian Bach, *Fugue, "The Little"*, BWV 578, c. 1707. Track 8 from a CBS Records CD titled Bach: Great Organ Favorites performed by E. Power Biggs. This is one of Bach's most popular fugues written for the organ. Recall that a fugue is a piece in which a main subject is presented successively in different voices (in this case different registers of the organ), sometimes coming back in inversion or retrograde or even retrograde-inversion. Meanwhile, a countersubject is often presented which complements the main subject heard at the outset. Often this countersubject is more active than the subject and it too may return in inversion or retrograde. Bach was arguably one of the greatest composers of fugues. Try and hear each of the different instances when the main subject reappears, noting that sometimes it shows up in a major key and sometimes in a minor key. The piece is written in G minor although it ends triumphantly on a major chord.

9. Johann Sebastian Bach, *Fugue No. 8 in D♯ minor* from *The Well-Tempered Clavier*, BWV 853, 1722. Track 16, Disc 1 from a Radio-Canada CD titled The Well-Tempered Clavier performed on harpsichord by Scott Ross. Bach wrote 48 preludes and fugues, one each for the 12 major keys and 12 minor keys, over a period of 22 years. The two volume work has been referred to as the "pianist's Old Testament." As the title indicates, Bachs *Well-Tempered Clavier* was intended to be performed on a keyboard that was well-tempered, despite the daunting challenge of playing preludes and fugues in all 24 major and minor keys. Many have mistakingly assumed that Bach wrote the massive work to champion equal-temperament as a superior tuning system. In fact, there is ample evidence to suggest that this was not the case. The music for the main subject and two of its **inversions** are shown in Figure 5.5 of the text. Figure 5.6 shows an augmented subject beginning in measure 62. Occasionally the subject appears in all three voices, but staggered, as in a round.

10. John Philip Sousa, *The Thunderer*, 1889. Track 10 from the CD Sousa's Greatest Hits performed by the United States Marine Band. The opening to this march is an easily discernible **inversion**. Note that this inversion is different than those typically used by Bach in that it happens *simultaneously* rather than delayed (see Figure 5.17 in the text).

11. Béla Bartók, *Mikrokosmos*, Book 6, BB 105: no. 141, *Subject and Reflection*, 1926–1939. Track 45 from a complete recording of this famous set of piano pieces, performed by Jenö Jandó (Naxos). The entire *Mikrokosmos* consists of 153 pieces for piano ranging in difficulty from beginner to advanced and featuring some interesting musical traits including the use of whole-tone scales, Hungarian folk tunes, syncopation, and modal harmonies. In Bartók's own words, *Mikrokosmos* "appears as a synthesis of all the musical and technical problems which were treated and in some case only partially solved in the previous piano works." This particular piece is a fine example of an **inversion** and as with the previous musical example, this inversion happens simultaneously (at least for most of the work) so that the musical symmetry is readily apparent in the score as well as

upon listening (see Figures 5.7 and 5.8 of the course text). The inversion is exact in the sense that the musical intervals are mirrored precisely (right-hand up a minor third means left-hand down a minor third), a compositional technique not harmonically accepted during Bach's time, but one that Bartók clearly intended as evidenced by the title (even the number is symmetric). The inversion is slightly offset in the second half of the piece, which can be described mathematically as a *glide reflection*. To watch the symmetry unfold on the piano keyboard, see the illustrative video by Alan Chan: `https://www.youtube.com/watch?v=GVwSzAqbWjg`

12. Franz Liszt, *Hungarian Rhapsody No. 2*, 1847. Track 2 from a Deutsche Grammophon CD titled Liszt: The 19 Hungarian Rhapsodies, played by pianist Roberto Szidon (1973). This popular piano solo contains numerous **transpositions** of catchy melodies, sometimes repeated an octave higher or lower, other times shifted by a perfect fifth. Near the end of the piece, during a brief transitional passage (at 7:30 of this particular recording), there is a challenging set of measures combining several symmetries together. Here we find **transpositions**, **inversions**, and **retrogrades** in just a few measures. As with the Sousa excerpt, the symmetry relationships are fairly easy to hear. The full rhapsody is a favorite of pianists and was used in many animated cartoons including the *Bugs Bunny* short "Rhapsody Rabbit" (Warner Brothers) and the *Tom and Jerry* episode "The Cat Concerto" (MGM), which won the 1946 Academy Award for best cartoon.

13. George Gershwin, *I Got Rhythm* (lyrics by Ira Gershwin), 1930. Track 6 off a Concord Records CD titled Someone to Watch Over Me, sung by Susannah McCorkle. This is a quaint intro track discussing the lyrics of the tune written by the composer's brother.

14. George Gershwin, *I Got Rhythm*, 1930. Track 7 off the previous CD. Gershwin, who grew up in Brooklyn, NY, the son of Russian-Jewish parents, wrote this popular tune in his musical *Girl Crazy*. The main opening melody is four measures long and features a simple **retrograde** at the end of bar two, but with an important shift in the rhythm to give it a jazzy feel. The form of the song is an AABA structure with the B section containing an **inversion** (more in the recording than in the score) as well as a **transposition**. Thus, this simple 32-measure song contains three of the four mathematical symmetries discussed in Section 5.1 of the text.