From Music to Mathematics: Exploring the Connections

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Continued Fractions

Handout (with Exercises) for Section 4.5.2

1 Playing around with π

The famous mathematical constant π is irrational, with a nonrepeating, nonterminating decimal expansion. To 40 decimal places, π is

$$\pi = 3.1415926535897932384626433832795028841972\ldots.$$

Since π is irrational, it *cannot* be written as the ratio of two integers. However, we could try to *approximate* π using rational numbers. How do we find good approximations to irrational numbers, particularly with small numerators and denominators?

Begin by writing π as

$$\pi = 3 + 0.1415926535897\ldots.$$

One approximation for π is thus three (this is actually inferred in the Bible), but that is not very impressive. We could continue by approximating $0.14\ldots$ as $14/100 = 7/50$ in order to obtain the first two decimal places of π. However, there is a more clever approach. Instead of approximating $0.14\ldots$ as $7/50$, consider inverting it twice, writing

$$0.1415926535897\ldots = \frac{1}{\dfrac{1}{0.1415926535897\ldots}} = \frac{1}{7.062513305931\ldots}.$$

Ignoring the decimals after $7.06\ldots$, this suggests the approximation $3 + 1/7 = 22/7$, a well-known estimate for π. As with the fraction $157/50 = 3 + 7/50$, this gives π to two decimal places, but notice that $22/7$ has a denominator about 7 times *smaller* than $157/50$. This is more in line with our goal of finding low-numbered rational approximations.

We can continue this process. Inverting the new "remainder" decimal $0.062513305931\ldots$ twice gives

$$0.062513305931\ldots = \frac{1}{\dfrac{1}{0.062513305931\ldots}} = \frac{1}{15.99659441\ldots}.$$

This last number in the denominator is very close to 16. If we approximate at this point, we find

$$\pi \approx 3 + \cfrac{1}{7 + \cfrac{1}{16}} = 3 + \frac{16}{113} = \frac{355}{113} = 3.14159292\ldots.$$

Remarkably, the approximation $355/113$ agrees with π to six decimal places! If we tried to obtain this approximation by truncating π to six decimal places, we would have a reduced fraction with denominator 125,000. That's roughly 1100 times larger than 113.

If we continue this process of inverting the remainder decimal, after two more iterations we obtain the following multilayered fraction:

$$\pi \approx 3+\cfrac{1}{7+\cfrac{1}{15+\cfrac{1}{1+\frac{1}{293}}}} = \frac{104{,}348}{33{,}215} = 3.14159265392142\ldots, \tag{1}$$

which approximates π accurately to nine decimal places.

These crazy, multistory fractions are called **continued fractions** and are an important topic in the subject of number theory, although they find their way into all sorts of applications in other fields.

2 Continued Fractions

Definition 2.1 *Given a real number α, the continued fraction expansion of α is*

$$\alpha = a_0+\cfrac{1}{a_1+\cfrac{1}{a_2+\cfrac{1}{a_3+\cfrac{1}{\ddots}}}} = [a_0; a_1, a_2, a_3, \ldots]$$

where each a_i (except possibly a_0) is a positive integer.

The notation $\alpha = [a_0; a_1, a_2, a_3, \ldots]$ is much easier to use than writing the multistory fraction. The first integer a_0 is just the number to the left of the decimal point (e.g., $a_0 = 3$ for $\alpha = \pi$). Notice the semicolon after the a_0. The succeeding list of integers are the denominators of each fraction since we know that each numerator is always 1. For example, from Equation (1), we have that

$$\pi = [3; 7, 15, 1, 292, \ldots].$$

The reason for 292, as opposed to 293, is that the final approximation in the denominator was $292.63459\ldots$. Although we rounded this to 293 to obtain the approximation 104,348/33,215 for π, the actual integer *before* rounding is 292.

Observe that if the number we are approximating is irrational, then this process of inverting the remainder decimal twice will continue forever; otherwise, the approximation would be exact and we would obtain a rational number.

Theorem 2.2 *The continued fraction expansion of a real number is finite if and only if that number is rational. In other words, the continued fraction expansion of an irrational number is infinite.*

Example 2.3 *Compute the continued fraction expansion for the rational number $\alpha = 37/13$.*

Solution: We start by dividing 37 by 13 to obtain 2 with a remainder of 11. Thus,

$$\frac{37}{13} = 2+\frac{11}{13} = 2+\frac{1}{13/11}.$$

Notice that the final fraction in the denominator is bigger than 1. This will always be the case since we are inverting a number less than 1 (the remainder). Continuing, we divide 13 by 11 to obtain 1 with a remainder of 2. Thus, we have

$$\frac{37}{13} \;=\; 2+\frac{11}{13} \;=\; 2+\frac{1}{13/11} \;=\; 2+\cfrac{1}{1+\cfrac{2}{11}} \;=\; 2+\cfrac{1}{1+\cfrac{1}{11/2}}\,.$$

Next, we divide 11 by 2 to obtain 5 with a remainder of 1. This gives

$$\frac{37}{13} \;=\; 2+\frac{11}{13} \;=\; 2+\frac{1}{13/11} \;=\; 2+\cfrac{1}{1+\cfrac{2}{11}} \;=\; 2+\cfrac{1}{1+\cfrac{1}{11/2}} \;=\; 2+\cfrac{1}{1+\cfrac{1}{5+\cfrac{1}{2}}}\,.$$

The process terminates at this point because we are left with the integer 2 in the denominator rather than a fraction. There is nothing left to divide when the remainder is 1. The last fraction of $11/2$ is simply $5+1/2$, which is already in the required form because of the 1 in the numerator. Thus, we have shown that the continued fraction expansion for the rational number $37/13$ is

$$\frac{37}{13} \;=\; [2;1,5,2]\,.$$

□

Note that before we reached the final step in the example, each of the previous steps did not have a 1 for a remainder. This is a useful fact.

Key Fact: When writing the finite continued fraction expansion of a rational number, the process terminates precisely when a remainder of 1 occurs.

2.1 Periodic Expansions

Sometimes, instead of terminating, the continued fraction expansion will repeat a certain pattern forever. For example,

$$\sqrt{8} \;=\; [2;1,4,1,4,1,4,1,4,\ldots]$$

has a repeating pattern of period 2 since it ends with the sequence $1,4,1,4,\ldots$. We will refer to this kind of sequence as a *periodic sequence of period 2.* When a continued fraction expansion has a periodic sequence, we write

$$\sqrt{8} \;=\; [2;\overline{1,4}\,].$$

Another example is

$$\sqrt{7} \;=\; [2;1,1,1,4,1,1,1,4,1,1,1,4,\ldots] \;=\; [2;\overline{1,1,1,4}\,].$$

Example 2.4 *Find the irrational number that corresponds to the continued fraction expansion*

$$\alpha \;=\; [1;2,2,2,\ldots] \;=\; [1;\overline{2}\,].$$

Solution: One way to approach the problem is to approximate α by terminating the expansion at different locations. These approximations are called **convergents**. In general, the

$$n^{\text{th}} \text{ convergent to } \alpha \;=\; \frac{p_n}{q_n} \;=\; [a_0; a_1, a_2, a_3, \ldots, a_n]\,.$$

The larger n is (the further out in the expansion we go), the better the approximation to α becomes.

For $\alpha = [1; 2, 2, 2, \ldots]$, the first five convergents are given below:

$$\begin{aligned}
\frac{p_0}{q_0} &= 1 \qquad (\text{this is just } a_0),\\
\frac{p_1}{q_1} &= 1 + \frac{1}{2} = \frac{3}{2} = 1.5,\\
\frac{p_2}{q_2} &= 1 + \cfrac{1}{2 + \cfrac{1}{2}} = \frac{7}{5} = 1.4,\\
\frac{p_3}{q_3} &= 1 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2}}} = \frac{17}{12} = 1.41\bar{6},\\
\frac{p_4}{q_4} &= 1 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2}}}} = \frac{41}{29} \approx 1.4138.
\end{aligned}$$

Do you recognize the last approximation of 1.414? This suggests that $\alpha = \sqrt{2}$, which turns out to be correct. Notice that not only are the convergents getting closer to $\sqrt{2}$, but they also oscillate about it (below, above, below, above, etc.). This will always be the case, no matter the value of α.

There is a clever trick to see that $\alpha = \sqrt{2}$. Since the continued fraction pattern is periodic, the expression for α actually *reappears* inside its own expansion. We have

$$\alpha \;=\; 1 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{\ddots}}}} \;=\; 1 + \cfrac{1}{1 + 1 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{\ddots}}}} \;=\; 1 + \frac{1}{1+\alpha}.$$

This last expression may seem strange, but if we know that the infinite continued fraction converges to a real number (this requires a proof), then the continued fraction expansion in the denominator of the penultimate expression is equal to the original continued fraction expansion for α. The main concept being invoked here is known as **self-similarity**, a key ingredient in the fascinating field of **chaos theory**.

Thus, we have that

$$\begin{aligned}
\alpha = 1 + \frac{1}{1+\alpha} \quad &\Longrightarrow \quad \alpha - 1 = \frac{1}{1+\alpha}\\
&\Longrightarrow \alpha^2 - 1 = 1 \qquad (\text{by cross-multiplying})\\
&\Longrightarrow \alpha^2 = 2\\
&\Longrightarrow \alpha = \sqrt{2}\,.
\end{aligned}$$

Note that we choose the positive square root because it is clear from the continued fraction expansion that $\alpha > 0$. □

This example is illustrative of an important fact about continued fractions.

Theorem 2.5 *The continued fraction expansion of an irrational number is periodic if and only if α is of the form $\alpha = r + s\sqrt{n}$, where r, s are rational numbers and n is a positive integer not equal to a perfect square. In this case, periodic means that the continued fraction expansion ends with a periodic, repeating sequence of positive integers.*

2.2 Convergents and Their Accuracy

Instead of having to compute the value of the convergents from scratch each time, there are some convenient **recursive formulas** available that give the next convergent in terms of the previous entries. The formulas are:

$$\begin{aligned} p_0 &= a_0 \\ p_1 &= a_1 a_0 + 1 \\ p_n &= a_n\, p_{n-1} + p_{n-2} \quad \text{if } n \geq 2, \end{aligned}$$

and

$$\begin{aligned} q_0 &= 1 \\ q_1 &= a_1 \\ q_n &= a_n\, q_{n-1} + q_{n-2} \quad \text{if } n \geq 2. \end{aligned}$$

For example, if $\alpha = \sqrt{2} = [1; 2, 2, 2, \ldots]$, then $p_0 = 1, p_1 = 2 \cdot 1 + 1 = 3$, and

$$\begin{aligned} p_2 &= a_2\, p_1 + p_0 = 2 \cdot 3 + 1 = 7 \\ p_3 &= a_3\, p_2 + p_1 = 2 \cdot 7 + 3 = 17 \\ p_4 &= a_4\, p_3 + p_2 = 2 \cdot 17 + 7 = 41. \end{aligned}$$

A similar set of calculations can be used to find the denominators q_n. Try checking them with $\sqrt{2}$ and comparing with the convergents we obtained in Example 2.4.

Before closing our discussion of continued fractions, we make one final point concerning the accuracy of the convergents. It turns out that the rational approximations to an irrational number α obtained by using a continued fraction expansion are the *best* possible for a given denominator. Moreover, each convergent is closer to α than the preceding one, and the convergents oscillate about α, with the even convergents ($n = 0, 2, 4, \ldots$) lying below α and the odd convergents ($n = 1, 3, 5, \ldots$) lying above. Specifically, we have the following theorem:

Theorem 2.6 *If $\{p_n/q_n\}$ is the sequence of convergents to an irrational number α, then*

$$\frac{p_n}{q_n} - \frac{p_{n-1}}{q_{n-1}} = \frac{(-1)^{n+1}}{q_{n-1}\, q_n}.$$

Moreover, any particular convergent p_n/q_n is closer to α than any other fraction whose denominator is less than q_n.

For example, when approximating $\alpha = \sqrt{2}$, choosing the fourth convergent $41/29$ gives the best possible rational approximation to $\sqrt{2}$ with denominator less than 29. This is precisely why the convergents of a continued fraction expansion are considered to be the *best* rational approximations.

3 Exercises

1. Compute the continued fraction expansion for the rational numbers $\alpha = 67/9$ and $\beta = 57/16$. Be sure to show your work. Give each answer in the form $[a_0; a_1, a_2, \ldots]$.

2. Compute the continued fraction expansion for the irrational numbers $\alpha = \sqrt{10}$ and $\beta = \sqrt{14}$. Be sure to show your work. Give each answer in the form $[a_0; a_1, a_2, \ldots]$. *Hint:* Each answer is periodic.

3. Consider the irrational number α with periodic continued fraction expansion $[1; 1, 1, 1, \ldots]$.

   a) Compute the first seven convergents of α, that is, compute p_n/q_n for $n = 0, 1, \ldots, 6$. What do you notice about the numbers in the numerator and denominator?

   b) Find the exact value of α using the method of self-similarity demonstrated in Example 2.4. What special name is given to the number α?

4. The number $\log_2(3/2)$ is important when attempting to find good approximations to a true or just perfect fifth.

   a) Compute the first seven convergents of $\log_2(3/2)$, that is, compute p_n/q_n for $n = 0, 1, \ldots, 6$. Be sure to show your work.

   b) If you have done part **a)** correctly, you should recognize the fraction obtained for $n = 4$. Explain how this particular fraction relates to equal temperament.

   c) Why would it "work" to divide the octave into 53 equal parts, creating a scale with 53 notes equally spaced by the step $H = 2^{1/53}$? In this case, what is the value of the frequency multiplier to raise the pitch by a perfect fifth? (This is the number that you would multiply the frequency of the tonic by in order to raise the pitch a P5.) What is the value of this multiplier in cents (to three decimal places) and how close is it (in cents, three decimal places) to a just perfect fifth?

   d) Consider the 53-tone equally tempered scale, where the distance between consecutive scale degrees is $2^{1/53}$. How many steps are in a major third? Convert the multiplier to raise the pitch a major third into cents (three decimal places). How close is it (in cents, three decimal places) to a just major third? Would you need to raise or lower the pitch in your new scale to obtain a just major third?

References

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[3] E. Weisstein, *Continued Fraction*, MathWorld: A Wolfram Web Resource, `http://mathworld.wolfram.com/ContinuedFraction.html`