From Music to Mathematics: Exploring the Connections

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The Fibonacci Numbers and the Golden Ratio

Worksheet for Sections 1.4 and/or 5.2

The **Fibonacci numbers** are the following infinite sequence of numbers:

$$1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, \ldots.$$

They are defined according to the following rules:

$$F_1 = 1, \quad F_2 = 1, \quad \text{and} \quad F_n = F_{n-1} + F_{n-2} \text{ for each integer } n \geq 3.$$

Here, F_n denotes the nth Fibonacci number, so $F_3 = 2, F_7 = 13$ and $F_{2016} = F_{2015} + F_{2014}$. We refer to n as an **index**, since it marks the place of the number in the sequence. The Fibonacci numbers were first described by the Indian scholar **Hemachandra** (1089–1172), who was studying rhythmic patterns in Sanskrit poetry (see Section 1.4 of the text).

A similar sequence of numbers is the **Lucas numbers**, defined with the same recursive relation, but with different initial **seeds** (starting values).

$$L_1 = 1, \quad L_2 = 3, \quad \text{and} \quad L_n = L_{n-1} + L_{n-2} \text{ for each integer } n \geq 3.$$

Here, L_n denotes the nth Lucas number. The Lucas numbers are closely tied to the Fibonacci numbers. There are identities between numbers within each sequence and identities relating numbers across the two sequences.

Related to these two sequences is the famous constant ϕ (pronounced "fee" or sometimes "fai"), known as the **golden ratio**. It has the value

$$\phi = \frac{1 + \sqrt{5}}{2} \approx 1.61803398875\,. \tag{1}$$

This constant is also called the **golden mean**, the **golden section**, or the **divine proportion**.

The golden ratio is defined as follows: Consider two quantities a and b and suppose that a is larger than b. If the ratio of the larger quantity to the smaller is the same as the ratio of the total sum to the larger, then the special ratio between the two numbers is called the **golden ratio** (see Figure 1). In other words, a/b is the golden ratio if the following equation holds true:

$$\frac{a}{b} = \frac{a+b}{a}\,. \tag{2}$$

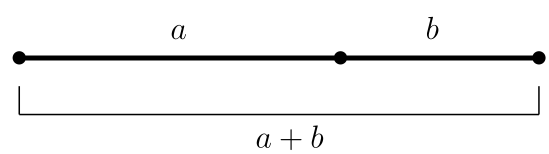

Figure 1: The ratio $a\!:\!b$ equals the ratio $a+b\!:\!a$, called the golden ratio.

It might seem that the golden ratio depends on the values of a and b. This is misleading. There is only one golden ratio. If we divide the top and bottom of the fraction on the right-hand side of equation (2) by b, we obtain

$$\frac{a}{b} = \frac{\frac{a}{b}+1}{\frac{a}{b}}. \tag{3}$$

This new equation is now written solely in terms of the quantity a/b, the quantity we seek. Setting $\phi = a/b$ in equation (3) yields an equation in one variable,

$$\phi = \frac{\phi+1}{\phi}, \tag{4}$$

which, after solving for ϕ, yields the value of the golden ratio given in equation (1).

Exercises:

1. Write out the first 13 Lucas numbers. Which sequence seems to be growing faster, Fibonacci or Lucas?

2. Solve equation (4) for ϕ to derive the value of the golden ratio.

3. Suppose a piece of music is 89 measures long. Where should the climax of the piece be located in order to divide it into two parts whose proportion equals the golden ratio? There are two answers. Find both of them, rounding to the nearest measure. What is significant about these numbers? Do the same problem for a piece of music 144 measures long.

4. **Ratios of Consecutive Fibonacci Numbers:**

   (a) Compute the ratios of consecutive Fibonacci numbers, starting with $\frac{F_2}{F_1}, \frac{F_3}{F_2}$, and continuing until you reach $\frac{F_{13}}{F_{12}}$. Where does this sequence of ratios seem to be heading? How does it approach this limit (from below, from above, or oscillating about the limit)?

   (b) Answer the same questions as part **(a)** using the Lucas numbers. Compare your results.

   (c) Finally, form your own sequence by using *any* two positive integers as your seed numbers, (e.g., $G_1 = 2$, $G_2 = 5$) and using the same recursive definition as the Fibonacci numbers to form subsequent terms ($G_n = G_{n-1} + G_{n-2}$). What do you observe about the ratios of consecutive terms in your sequence? Make a bold, conclusive statement (i.e., theorem) based on your answers to these questions.

5. **Fibonacci Identities:** Compute each of the sums below in several cases, starting with the case $n = 1$, then the case $n = 2$, and so on. When you think you have figured out a pattern for the general sum, guess the other side of the identity. A typical answer would look something like $F_{n+1} + 2$. Then check your guess against some larger choices for n to see if it holds up.

   (a) $F_1 + F_3 + \cdots + F_{2n-1} =$ ____________________.

   (b) $F_1 + F_2 + \cdots + F_n =$ ____________________.

   (c) $F_n F_{n+2} - F_{n+1}^2 =$ ____________________.

   (d) $F_1^2 + F_2^2 + \cdots + F_n^2 =$ ____________________.

6. **Some Surprising Divisibility Properties:**

(a) Find instances where one Fibonacci number is a factor (divisor) of another Fibonacci number (see the table below). What is the relationship between their two indices? Make a bold statement based on your findings. Does this same result work with the Lucas numbers?

(b) Compute the greatest common divisor for different pairs of Fibonacci numbers (see table below). What do you notice about the number that results? Try this with Lucas numbers as well. Once again, make a conclusive statement based on your results.

Some Fibonacci Numbers F_n, Lucas Numbers L_n, and their prime factorizations:

$F_1 = 1$	$L_1 = 1$
$F_2 = 1$	$L_2 = 3$
$F_3 = 2$	$L_3 = 4 = 2^2$
$F_4 = 3$	$L_4 = 7$
$F_5 = 5$	$L_5 = 11$
$F_6 = 8 = 2^3$	$L_6 = 18 = 2 \cdot 3^2$
$F_7 = 13$	$L_7 = 29$
$F_8 = 21 = 3 \cdot 7$	$L_8 = 47$
$F_9 = 34 = 2 \cdot 17$	$L_9 = 76 = 2^2 \cdot 19$
$F_{10} = 55 = 5 \cdot 11$	$L_{10} = 123 = 3 \cdot 41$
$F_{11} = 89$	$L_{11} = 199$
$F_{12} = 144 = 2^4 \cdot 3^2$	$L_{12} = 322 = 2 \cdot 7 \cdot 43$
$F_{13} = 233$	$L_{13} = 521$
$F_{14} = 377 = 13 \cdot 29$	$L_{14} = 843 = 3 \cdot 281$
$F_{15} = 610 = 2 \cdot 5 \cdot 61$	$L_{15} = 1364 = 2^2 \cdot 11 \cdot 31$
$F_{16} = 987 = 3 \cdot 7 \cdot 47$	$L_{16} = 2207$
$F_{17} = 1597$	$L_{17} = 3571$
$F_{18} = 2584 = 2^3 \cdot 17 \cdot 19$	$L_{18} = 5778 = 2 \cdot 3^3 \cdot 107$
$F_{19} = 4181 = 37 \cdot 113$	$L_{19} = 9349$
$F_{20} = 6765 = 3 \cdot 5 \cdot 11 \cdot 41$	$L_{20} = 15127 = 7 \cdot 2161$
$F_{21} = 10946 = 2 \cdot 13 \cdot 421$	$L_{21} = 24476 = 2^2 \cdot 29 \cdot 211$
$F_{22} = 17711 = 89 \cdot 199$	$L_{22} = 39603 = 3 \cdot 43 \cdot 307$
$F_{23} = 28657$	$L_{23} = 64079 = 139 \cdot 461$
$F_{24} = 46368 = 2^5 \cdot 3^2 \cdot 7 \cdot 23$	$L_{24} = 103682 = 2 \cdot 47 \cdot 1103$
$F_{25} = 75025 = 5^2 \cdot 3001$	$L_{25} = 167761 = 11 \cdot 101 \cdot 151$
$F_{26} = 121393 = 233 \cdot 521$	$L_{26} = 271443 = 3 \cdot 90481$
$F_{27} = 196418 = 2 \cdot 17 \cdot 53 \cdot 109$	$L_{27} = 439204 = 2^2 \cdot 19 \cdot 5779$
$F_{28} = 317811 = 3 \cdot 13 \cdot 29 \cdot 281$	$L_{28} = 710647 = 7^2 \cdot 14503$
$F_{29} = 514229$	$L_{29} = 1149851 = 59 \cdot 19489$
$F_{30} = 832040 = 2^3 \cdot 5 \cdot 11 \cdot 31 \cdot 61$	$L_{30} = 1860498 = 2 \cdot 3^2 \cdot 41 \cdot 2521$