From Music to Mathematics: Exploring the Connections

Gareth E. Roberts

The Monochord Lab: Length Versus Pitch

Names: ____

Note for instructors: Since the point of this project is to derive (and hear) the Pythagorean scale, this lab should be taught before proceeding to the material in Chapter 4. The lab requires the use of a monochord (a one-stringed instrument) containing a movable fret that easily allows different string lengths to be plucked. Monochords vary in prices depending on quality. The company Artec Educational sells a kit to construct a monochord for under $15. It is also important to have a ruler running parallel to the string to determine string lengths. Students can work in small groups, with one monochord per group. The project has two parts; the second part should not be started (or viewed) until the first part is completed correctly.

The goal of this lab project is for you to explore the relationship between the length of a string and the pitch sounded when the string is plucked. The device you will use to investigate this relationship is called a **monochord**. You will recreate a famous and ancient musical scale known as the **Pythagorean scale**. You should work in a group of three to four people (one monochord per group) answering questions and filling in your results as you complete the lab. Please turn in one lab report per group, listing the names of all the members at the top of the first page. After the lab is complete, we will perform some simple pieces of music as a class. Before beginning the lab, please read the safety instructions below.

Safety

- The end of the string is sharp. Be careful when handling the monochord near the tuner end of the instrument.
- Over-tightening the string may cause it to break, allowing the two broken pieces to fly around somewhat. The broken ends will be sharp, so handle any broken string carefully. Keep your face away from the string so that you do not get hit in the eye by a breaking string.
- This instrument is not for "twanging" as rock guitarists play their instruments. This type of playing may generate a louder sound, but the string life is reduced significantly. Sufficient volume may be generated with just a gentle "pluck" of the string.

The Pythagorean Scale

As legend goes, the great Greek mathematician and philosopher Pythagoras was walking by a blacksmith's shop when he noticed that certain sounds of the hammers on the anvils were more consonant than others. Upon investigation, he discovered that the nicer sounds came from hammers whose weights were in simple proportions like $2:1$, $3:2$, and $4:3$. After further musical experiments with strings and lutes, he and his followers (the Pythagoreans) became convinced that basic musical harmony could be expressed through simple ratios of whole numbers. Their general belief was that the lower the numbers in the ratio, the better the notes sounded together. Thus, the natural building blocks of mathematics (small whole numbers) were aligned with the natural interval relationships used to create harmonious music. This helps explain the devotion of the Pythagoreans to rational numbers, numbers that can be expressed as the ratio of two integers.

Using the monochord, you will investigate this Pythagorean belief and find the ratios used in the oldest musical scale, the Pythagorean scale. The goal is to investigate the relationship between the length of the string vibrating and the change in pitch (musical interval) produced.

Part I: Simple Ratios

1. The tension T of the string may be adjusted by tightening the peg at one end of the monochord. What happens to the pitch as the string is tightened?

2. Place the sliding fret at exactly half the length of the string. Hold one finger over the fret and pluck the string gently. What is the relationship between the pitch of this note versus the note produced without the fret? If you are not sure about the relationship, you can try finding the notes on a piano, although first you will have to "tune" the monochord (on the open, unfretted string) to a note on the piano.

3. Place the sliding fret at exactly 1/4 of the string's length. Hold the string down over the fret and pluck the smaller portion of the string. What is the musical relationship between this note and the one sounded by plucking half the string? What is the relationship between this note and the one sounded on the open string?

4. Place the sliding fret at exactly 1/3 of the string's length. Hold the string down over the fret and pluck the two pieces to either side of the fret. What is the musical interval between these two pitches? Draw an important conclusion:

   Cutting the length of the string in **half** ________________ (raises or lowers) the pitch by ________________ (what musical interval?).

5. Next, investigate what happens if the ratio of the string lengths is 2 : 3. In other words, what effect does changing the string length to 2/3 its original value have on the pitch?

6. Finally, investigate what happens if the ratio of the string lengths is 3 : 4. In other words, what effect does changing the string length to 3/4 its original value have on the pitch? Given that $\frac{3}{4} = \frac{1}{2} \cdot \frac{3}{2}$, how could this result have been predicted using the answers to the previous two questions?

Before proceeding to the next part of the lab, check with your instructor to make sure that you have answered the questions above correctly.

Part 2: Ratios for the Pythagorean Scale

Given the ground work accomplished above, we are now ready to construct the entire Pythagorean scale. Although it sounds like the usual major scale, it differs in some important ways from the scale found on a modern piano. These differences are explored in great detail in Chapter 4.

Thus far, we have uncovered three important facts concerning the relationship between the ratios of string lengths and the corresponding musical interval.

1. String lengths in a ratio of $1:2$ are an octave apart. Specifically, cutting the length of the string in half raises the pitch an octave. Conversely, doubling the length of the string lowers the pitch an octave.

2. String lengths in a ratio of $2:3$ are a perfect fifth apart. Specifically, cutting the length of the string by a factor of $2/3$ raises the pitch by a perfect fifth.

3. String lengths in a ratio of $3:4$ are a perfect fourth apart. Specifically, cutting the length of the string by a factor of $3/4$ raises the pitch by a perfect fourth.

Notice the musical and mathematical simplicity of these facts. Simple ratios lead to the "perfect" intervals of an octave, fifth, and fourth. It is no accident that these are the primary intervals of early music (e.g., Gregorian chant used octaves, while Medieval polyphony used fourths and fifths). They also underscore the critical tonic-dominant harmonic relationship and the V–I and I–IV–V chord progressions commonly found in music.

It is also important to recognize that fact 3 can be derived from the first two facts. Multiplying the length of the string by $1/2$ raises the pitch an octave. Then, multiplying the length of the new string by a factor of $3/2$ will *lower* the pitch by a perfect fifth (longer strings have lower pitches). Going up an octave and then down by a perfect fifth is equivalent to going up by a perfect fourth (e.g., middle C up an octave to the next highest C and then down a perfect fifth gives F, and the interval between middle C and this F is a perfect fourth). Since

$$\frac{1}{2} \cdot \frac{3}{2} = \frac{3}{4}, \tag{1}$$

the net effect of shortening the string length by a factor of $3/4$ is to raise the pitch by a perfect fourth. Notice that we multiply the two ratios in Equation (1), as opposed to adding them. *We are always interested in the ratios of string lengths, not their difference!*

The goal for Part 2 of this lab is to discover the remaining ratios in the Pythagorean scale. The scale is essentially the same as the major scale, except the string lengths are slightly different from the modern ones. We will derive this scale using only facts 1 and 2 above and the circle of fifths (see Section 2.4 of the text). For example, to find the second note of the major scale, we go up two perfect fifths from the tonic and then down an octave. The net effect is raising the pitch by a whole step. If the starting note is middle C, then up a perfect fifth gives G, and up another perfect fifth gives D′ (the first two "hours" of the circle of fifths). We then go down an octave to find the note D just above the starting C. Using the ratios given in facts 1 and 2, what fraction of the full string do we take to obtain the second note of the scale? Use a calculator to find the actual length of the string (round to two decimal places) and play it with the open string to hear the first two notes of the Pythagorean scale (Do and Re).

The next note to find would be the sixth of the scale since that is a perfect fifth above the second. From there we can find the third and finally the seventh note of the scale, the leading tone. Make sure that each note is obtained using only the ratios from facts 1 and 2. Complete Table 1 and then try playing the full scale, perhaps as a class. Do you hear any differences between your scale and the major scale on a modern instrument?

Scale Degree	**Solfège Syllable**	**Interval**	**Ratio**	**Length**
1	Do	Unison	$\frac{1}{1} \cdot L$	
2	Re	Major Second		
3	Mi	Major Third		
4	Fa	Perfect Fourth	$\frac{3}{4} \cdot L$	
5	Sol	Perfect Fifth	$\frac{2}{3} \cdot L$	
6	La	Major Sixth		
7	Ti	Major Seventh		
1	Do	Octave	$\frac{1}{2} \cdot L$	

Table 1: The lengths and ratios of the Pythagorean scale. L represents the length of the full string. Give lengths in the last column to two decimal places.

Some Concluding Questions

1. Find the prime factorization of the numerator and denominator in each ratio in the fourth column of Table 1. For example, $3/4 = 3/2^2$. What do you notice about each factorization?

2. According to Table 1, what is the ratio of string lengths that are a whole step apart? Be sure to check **all** five whole steps in the scale (they should be identical). This ratio, call it W, is the factor used to shorten the string length in order to raise the pitch by a whole step. Express W as a ratio of two integers.

3. According to Table 1, what is the ratio of string lengths that are a half step apart? Be sure to check **both** half steps in the scale (they should be identical). This ratio, call it H, is the factor used to shorten the string length in order to raise the pitch by a half step. Express H as a ratio of two integers.

4. Recall that two half steps are equivalent to one whole step. Hence, it should be the case that $H \cdot H = H^2 = W$. Is this equation valid? Compute the quantity H^2/W, giving your answer as a ratio of two integers and in decimal form (to five decimal places). Also give the prime factorization of the numerator and denominator. This value is important in the theory of tuning and is known as the *Pythagorean comma*. This comma, as well as the problems it creates, is discussed in Section 4.1 of the text.