

From Music to Mathematics: Exploring the Connections

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Comparing the Three Musical Tuning Systems

Handout for Chapter 4

Scale Degree	Solfège	Interval	Pythagorean	Just Intonation	Equal Temp.
1	Do	Uni.	1	1	1
2	Re	M2	$\frac{9}{8} = 1.125$	$\frac{9}{8} = 1.125$	$2^{2/12} \approx 1.1225$
3	Mi	M3	$\frac{81}{64} = 1.265625$	$\frac{5}{4} = 1.25$	$2^{4/12} \approx 1.2599$
4	Fa	P4	$\frac{4}{3} = 1.3\bar{3}$	$\frac{4}{3} = 1.3\bar{3}$	$2^{5/12} \approx 1.3348$
5	Sol	P5	$\frac{3}{2} = 1.5$	$\frac{3}{2} = 1.5$	$2^{7/12} \approx 1.4983$
6	La	M6	$\frac{27}{16} = 1.6875$	$\frac{5}{3} = 1.6\bar{6}$	$2^{9/12} \approx 1.6818$
7	Ti	M7	$\frac{243}{128} = 1.8984375$	$\frac{15}{8} = 1.875$	$2^{11/12} \approx 1.8877$
8 = 1	Do	Oct.	2	2	2

Table 1: The ratios or multipliers used to *raise* a note (increase the frequency) by a major interval in the three different tuning systems discussed in Sections 4.1–4.3: Pythagorean scale, just intonation, and equal temperament. Pythagorean tuning and just intonation use rational numbers while equal temperament uses irrational multipliers (except for unison or the 2:1 octave).

Example 0.1 Find the frequency of the note C^\sharp above $A440$ Hz in each of the three different tuning systems.

Solution: Since C^\sharp is a major third above $A440$, we use the multipliers for scale degree 3 listed in Table 1. For the Pythagorean scale, we multiply 440 by 1.265625 to find that C^\sharp is 556.875 Hz. Using just intonation, we should tune C^\sharp to $440 \cdot 1.25 = 550$ Hz. Finally, in equal temperament, C^\sharp is given by $440 \cdot 2^{4/12} \approx 554.365$ Hz. \square

Example 0.2 Find the frequency of the note F just below $A440$ Hz in each of the three different tuning systems.

Solution: In this case, since the note F is a major third *below* $A440$, we must *divide* 440 by the multipliers for scale degree 3 listed in Table 1. For the Pythagorean scale, we calculate that the F just below $A440$ is $440 \div 1.1265625 \approx 347.654$ Hz. Using just intonation, we tune the F just below $A440$ to $440 \div 1.25 = 352$ Hz. Finally, in equal temperament, the F just below $A440$ is $440 \div 2^{4/12} \approx 349.228$ Hz. \square

Scale Degree	Solfège	Interval	Pythagorean	Just Intonation	Equal Temp.
1	Do	Uni.	0	0	0
2	Re	M2	203.9	203.9	200
3	Mi	M3	407.8	386.3	400
4	Fa	P4	498.0	498.0	500
5	Sol	P5	702.0	702.0	700
6	La	M6	905.9	884.4	900
7	Ti	M7	1109.8	1088.3	1100
8 = 1	Do	Oct.	1200	1200	1200

Table 2: A comparison of the three tuning systems using *cents*, rounded to one decimal place. Note that equal temperament does a good job approximating a perfect fifth (only 2 cents off), but is noticeably sharp (nearly 14 cents) of a just major third.

Cents were introduced by the mathematician Alexander Ellis (1804–90) around 1880. They are now a commonly used unit of measurement when comparing different tuning systems, or discussing non-traditional tunings. A typical listener can distinguish pitches that are between 4 and 8 cents apart. Cents are based on a logarithmic scale (like decibels). The formula for converting a ratio or multiplier r into cents is

$$\boxed{\# \text{ of cents} = 1200 \log_2(r) = 1200 \cdot \frac{\ln r}{\ln 2}.} \quad (1)$$

For example, a half step in equal temperament is given by the multiplier $2^{1/12}$. In cents, using the definition of the logarithm, this is

$$1200 \log_2(2^{1/12}) = 1200 \cdot \frac{1}{12} = 100 \text{ cents.}$$

Since all half steps are equal in equal temperament, one can easily obtain any interval in cents just by multiplying the number of half steps in the interval by 100 (see Table 2). For comparison, the Pythagorean comma is approximately 23.5 cents while the syntonic comma is roughly 21.5 cents.

Example 0.3 *Compute the number of cents in a major third using Pythagorean tuning and just intonation.*

Solution: In the Pythagorean scale, the ratio used to raise a pitch by a major third is $r = 81/64$. To convert this ratio into cents, we apply equation (1) to find that the number of cents in a Pythagorean major third is

$$1200 \cdot \frac{\ln(81/64)}{\ln 2} \approx 407.8 \text{ cents.}$$

In just intonation, the ratio for a major third is $5/4$, so we have

$$1200 \cdot \frac{\ln(5/4)}{\ln 2} \approx 386.3 \text{ cents}$$

in a just major third. Note the discrepancy between these values and the equally tempered major third of 400 cents. \square